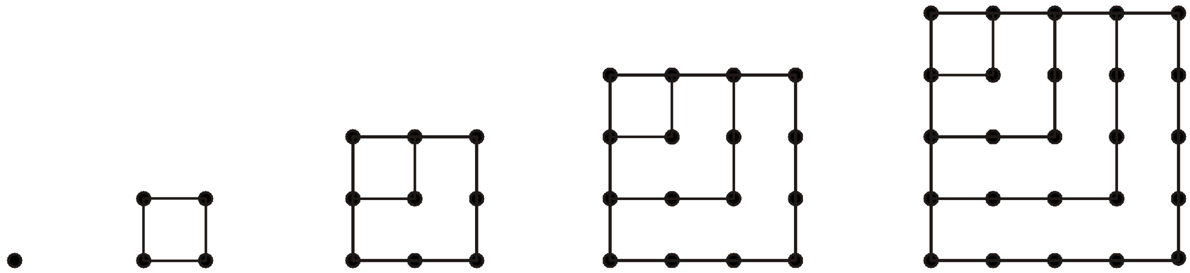
Dreieckzahlen, Quadratzahlen, …

Quadratzahlen Q1, Q2, Q3, … Qn



$Q1 = 1^2 = 1$	$Q2 = 2^2 = 4$	$Q3 = 3^2 = 9$	$Q4 = 4^2 = 16$	$Q5 = 5^2 = 25$
	$Q2 = Q1 + 3$	$Q3 = Q2 + 5$	$Q4 = Q3 + 7$	$Q5 = Q4 + 9$

Rekursionsformel: $Q(n + 1) = Qn + n + (n+1) = Qn + (2n + 1)$

Formel zur Berechnung von Qn: $Qn = n^2$

Aus Quadratzahlen lassen sich in einfacher Weise **„pythagoreische Zahlentripel"** ableiten. Unter einem pythagoreischen Zahlentripel versteht man drei natürliche Zahlen a, b und c für die gilt: $a^2 + b^2 = c^2$.

Es gilt ja allgemein: $Q(n + 1) = Qn + (2n + 1)$

Q(n + 1) und Qn sind ja Quadratzahlen. Wenn nun (2n + 1) auch eine Quadratzahl ist, so hat man ein pythagoreisches Zahlentripel. (2n + 1) muss eine **ungerade Quadratzahl** sein.

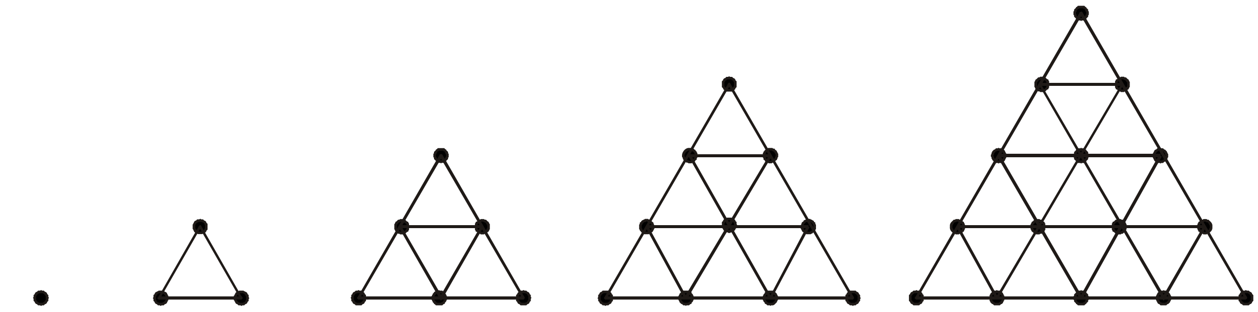
$2n + 1 = 9 = 3^2 \rightarrow n = 4 \rightarrow Q5 = Q4 + 3^2 \rightarrow 5^2 = 4^2 + 3^2 \rightarrow$ pythagoreisches Zahlentripel 3, 4, 5

$2n + 1 = 25 = 5^2 \rightarrow n = 12 \rightarrow Q13 = Q12 + 5^2 \rightarrow 13^2 = 12^2 + 5^2 \rightarrow$ pythagoreisches Zahlentripel 5, 12, 13

$2n + 1 = 49 = 7^2 \rightarrow n = 24 \rightarrow Q25 = Q24 + 7^2 \rightarrow 25^2 = 24^2 + 7^2 \rightarrow$ pythagoreisches Zahlentripel 7, 24, 25

$2n + 1 = 81 = 9^2 \rightarrow n = 40 \rightarrow Q41 = Q40 + 9^2 \rightarrow 41^2 = 40^2 + 9^2 \rightarrow$ pythagoreisches Zahlentripel 9, 40, 41

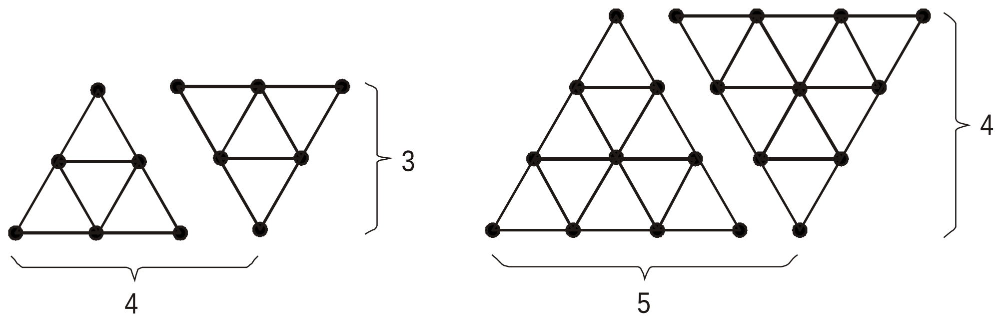
Dreieckzahlen D1, D2, D3, … Dn



D1 = 1 D2 = D1 + 2 = 3 D3 = D2 + 3 = 6 D4 = D3 + 4 = 10 D5 = D4 + 5 = 15

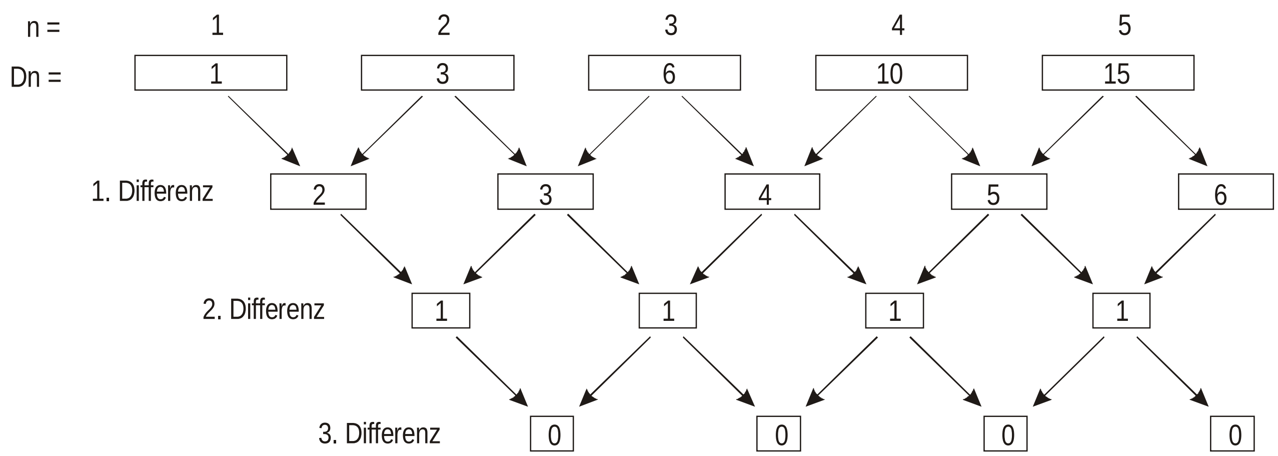
Rekursionsformel: D(n + 1) = Dn + (n + 1)

Formel zur Berechnung von Dn:

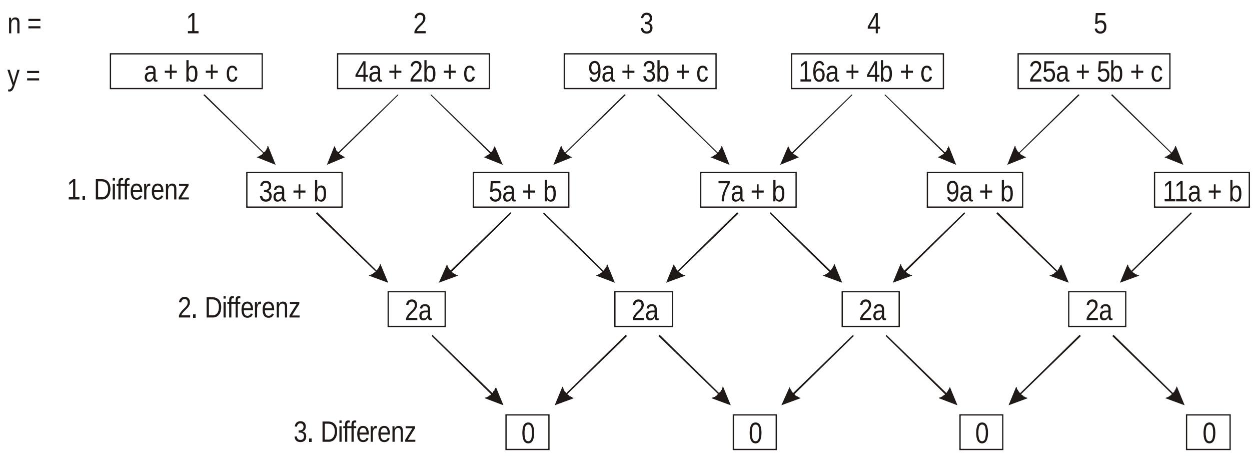


$D3 = \frac{3 \cdot 4}{2} = 6$ $D4 = \frac{4 \cdot 5}{2} = 10$ $Dn = \frac{n(n+1)}{2}$

Ein allgemeines Verfahren zur Berechnung von Dn:



$y = an^2 + bn + c$



Man erkennt: $1 = 2a \rightarrow a = \frac{1}{2}$

$2 = 3a + b = \frac{3}{2} + b \rightarrow b = \frac{1}{2}$

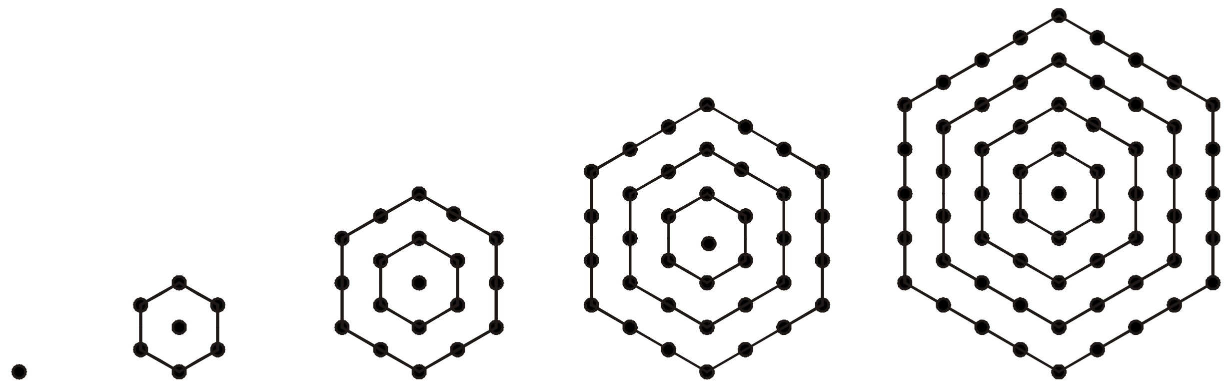
$1 = a + b + c = \frac{1}{2} + \frac{1}{2} + c \rightarrow c = 0$

Ergebnis: $Dn = \frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1) = \frac{n(n+1)}{2}$

Test: $D1 = \frac{1(1+1)}{2} = 1$, $D2 = \frac{2(2+1)}{2} = 3$, $D3 = \frac{3(3+1)}{2} = 6$, $D4 = \frac{4(4+1)}{2} = 10$, $D5 = \frac{5(5+1)}{2} = 15$, …

Kurioses über Dreieckzahlen: $D2^2 - D1^2 = 8 = 2^3$, $D3^2 - D2^2 = 27 = 3^3$, $D4^2 - D3^2 = 63 = 4^3$, … $Dn^2 - D(n-1)^2 = n^3$
Bitte nachprüfen!

Sechseckzahlen S1, S2, S3, … Sn

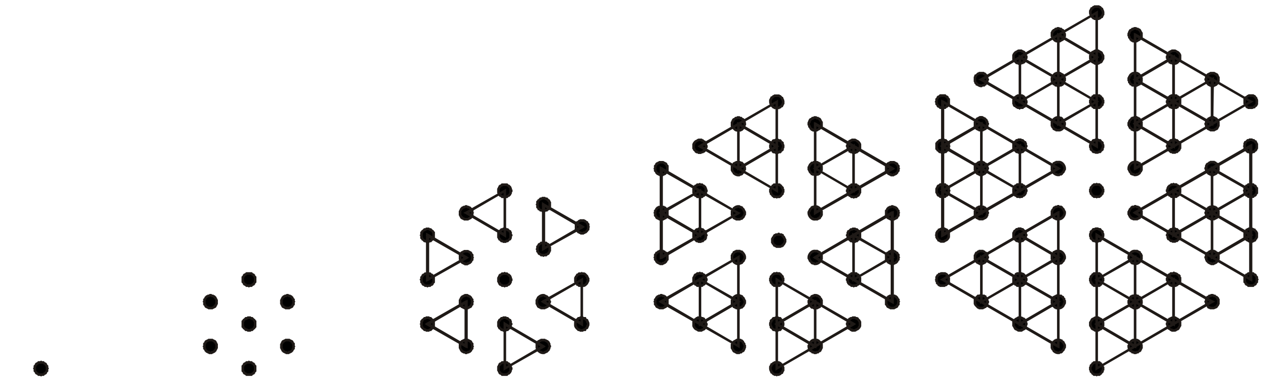


S1 = 1 S2 = S1 + 6 = 7 S3 = S2 + 2·6 = 19 S4 = S3 + 3·6 = 37 S5 = S4 + 4·6 = 61

Rekursionsformel: $S(n+1) = Sn + 6n$

Formel zur Berechnung von Sn:

Sechseckzahlen können aus Dreieckzahlen aufgebaut werden:



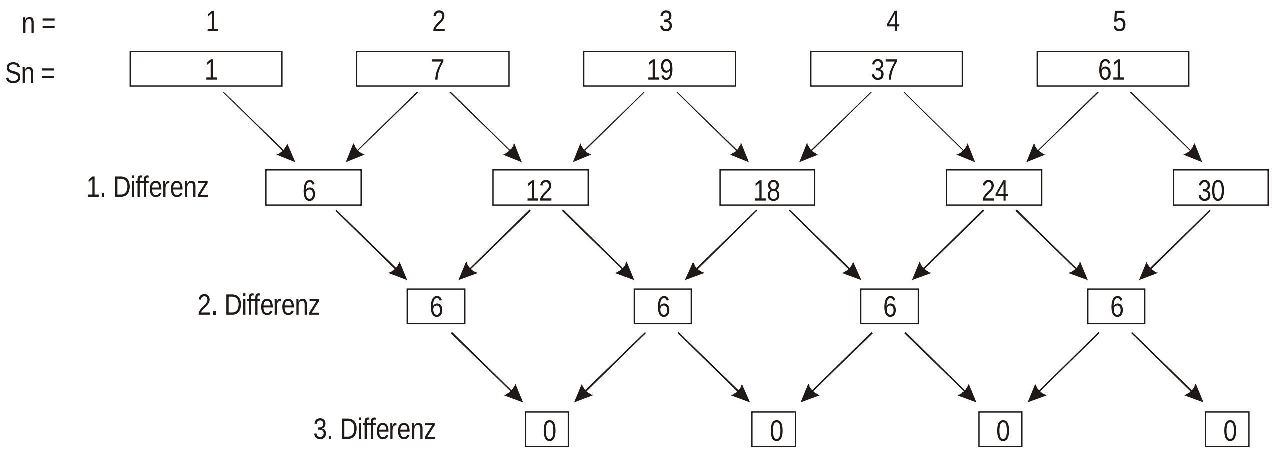
S1 = 1 | S2 = 6·D1 + 1 = 6·1 + 1 = 7 | S3 = 6·D2 + 1 = 6·3 + 1 = 19 | S4 = 6·D3 + 1 = 6·6 + 1 = 37 | S5 = 6·D4 + 1 = 6·10 + 1 = 61

Es gilt: $Dn = \frac{n(n+1)}{2} \rightarrow D(n-1) = \frac{(n-1)n}{2}$

Allgemein: $Sn = 6 \cdot D(n-1) + 1 = 6 \cdot \frac{(n-1)n}{2} + 1 = 3(n-1)n + 1 = 3n^2 - 3n + 1$

Test:
$S1 = 3 \cdot 1^2 - 3 \cdot 1 + 1 = 1$
$S2 = 3 \cdot 2^2 - 3 \cdot 2 + 1 = 7$
$S3 = 3 \cdot 3^2 - 3 \cdot 3 + 1 = 19$
$S4 = 3 \cdot 4^2 - 3 \cdot 4 + 1 = 37$
$S5 = 3 \cdot 5^2 - 3 \cdot 5 + 1 = 61$
$S6 = 3 \cdot 6^2 - 3 \cdot 6 + 1 = 91$
…

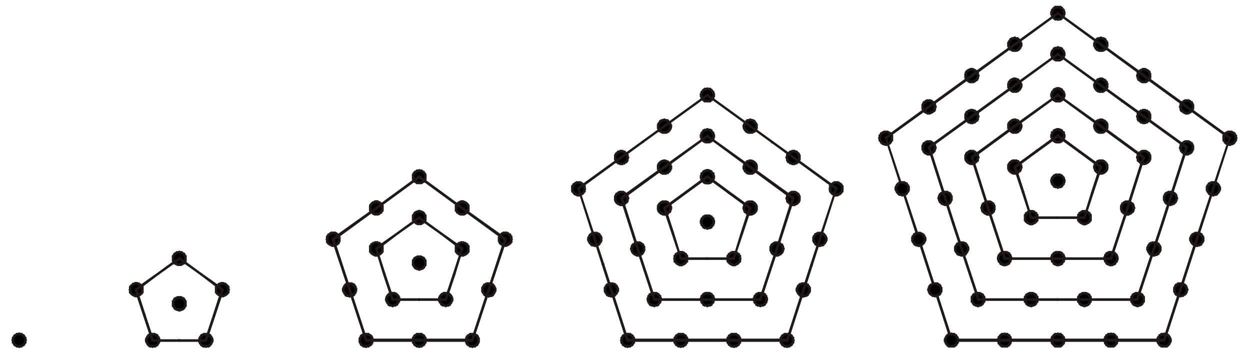
Zweite Art der Berechnung von Sn:



Man erkennt:
$6 = 2a \rightarrow a = 3$
$6 = 3a + b = 9 + b \rightarrow b = -3$
$1 = a + b + c = 3 - 3 + c \rightarrow c = 1$

Ergebnis: $Sn = 3n^2 - 3n + 1$

Fünfeckzahlen F1, F2, F3, … Fn



F1 = 1 F2 = F1 + 5 = 6 F3 = F2 + 2·5 = 16 F4 = F3 + 3·5 = 31 F5 = F4 + 4·5 = 51

Rekursionsformel: $F(n + 1) = Fn + 5n$

Berechnung von Fn:

n =	1	2	3	4	5
Fn =	1	6	16	31	51
1. Differenz	5	10	15	20	25
2. Differenz	5	5	5	5	
3. Differenz	0	0	0	0	

Man erkennt:

$5 = 2a \quad \rightarrow \quad a = \frac{5}{2}$

$5 = 3a + b = \frac{15}{2} + b \quad \rightarrow \quad b = -\frac{5}{2}$

$1 = a + b + c = \frac{5}{2} - \frac{5}{2} + c \quad \rightarrow \quad c = 1$

Ergebnis: $Fn = \frac{5}{2}n^2 - \frac{5}{2}n + 1 = \frac{5}{2}n(n - 1) + 1$

Test:

$F1 = \frac{5}{2}\cdot 1\cdot(1 - 1) + 1 = 1$

$F2 = \frac{5}{2}\cdot 2\cdot(2 - 1) + 1 = 6$

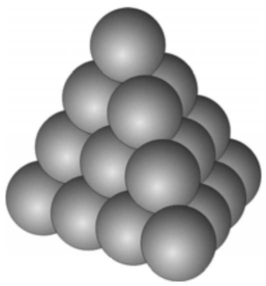
$F3 = \frac{5}{2}\cdot 3\cdot(3 - 1) + 1 = 16$

$F4 = \frac{5}{2}\cdot 4\cdot(4 - 1) + 1 = 31$

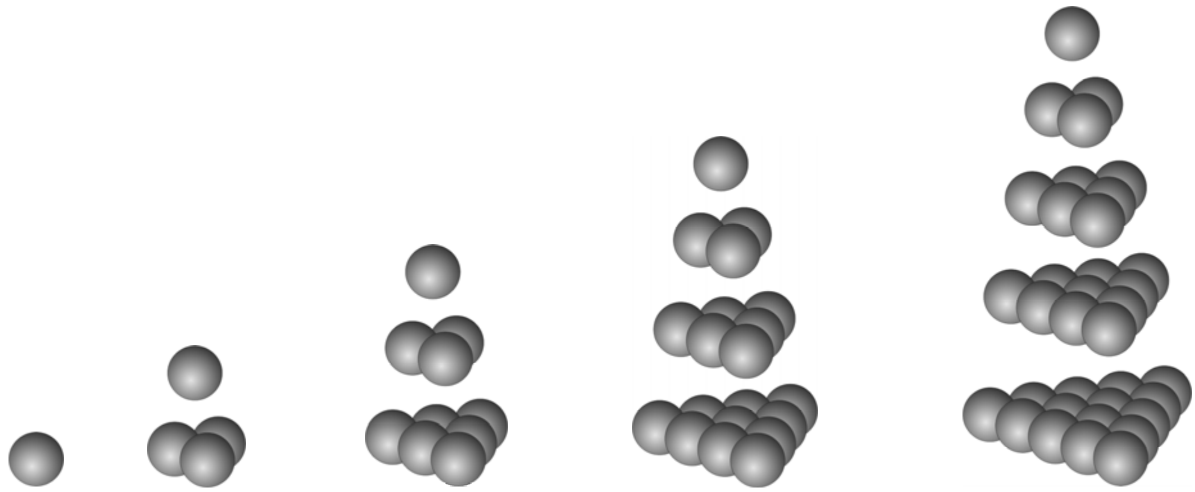
$F5 = \frac{5}{2}\cdot 5\cdot(5 - 1) + 1 = 51$

$F6 = \frac{5}{2}\cdot 6\cdot(6 - 1) + 1 = 76$

…



Tetraederzahlen T1, T2, T3, … Tn



T1 = 1

T2 = T1 + D2 = = 1 + 3 = 4

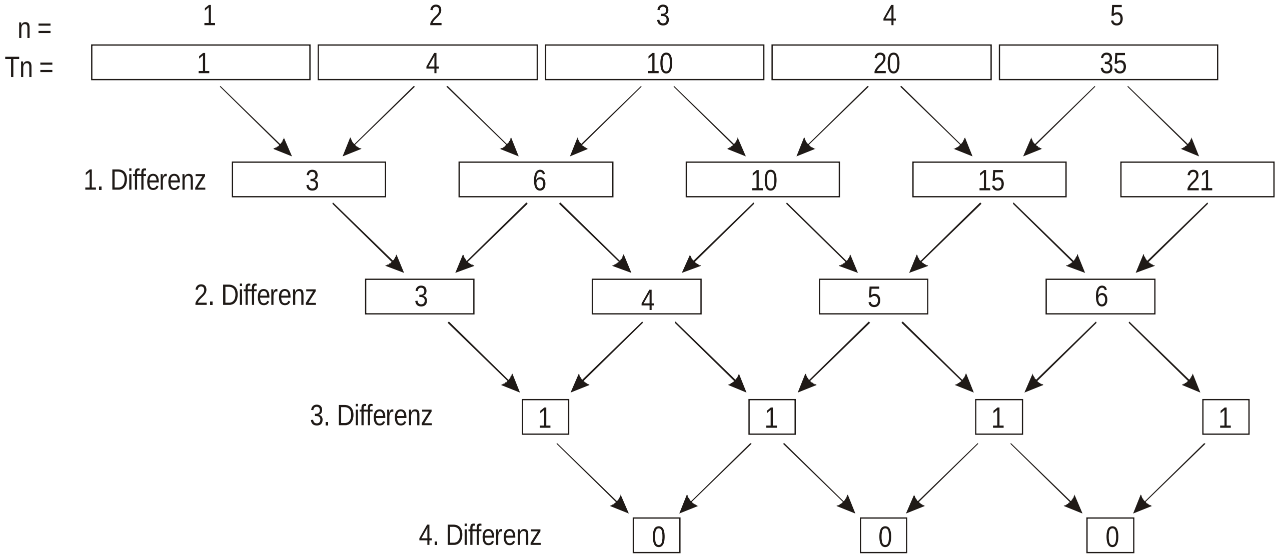
T3 = T2 + D3 = = 4 + 6 = 10

T4 = T3 + D4 = = 10 + 10 = 20

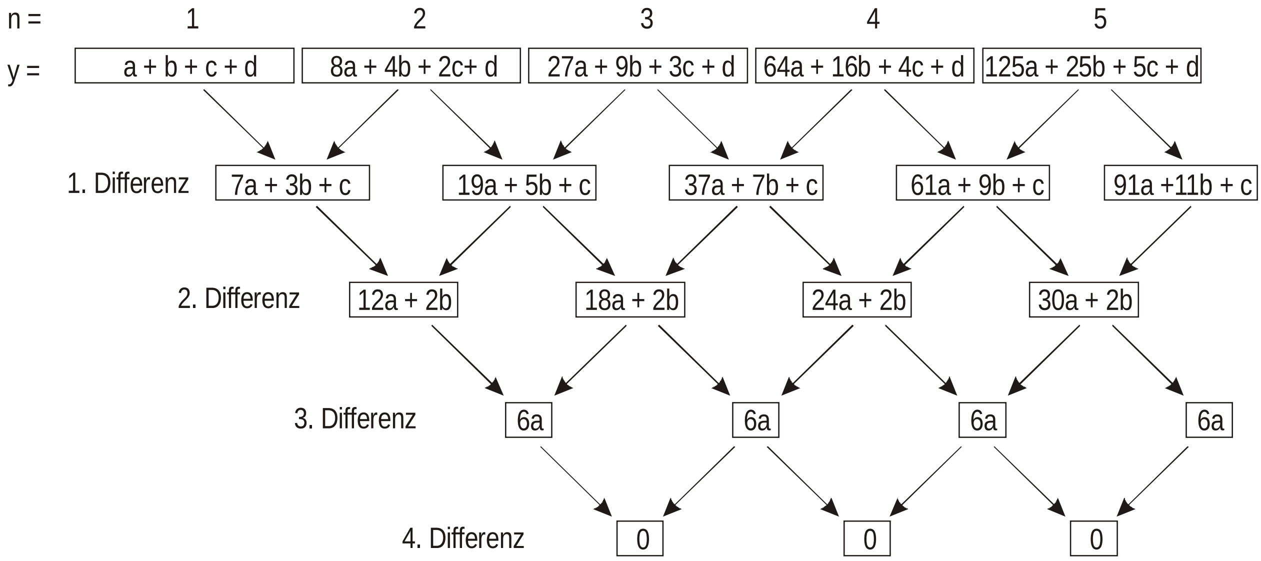
T5 = T4 + D5 = = 20 + 15 = 35

Rekursionsformel: $T(n + 1) = Tn + D(n + 1) = Tn + \frac{n(n+1)}{2}$

Berechnung von Tn:



$y = an^3 + bn^2 + cn + d$



Man erkennt:

$1 = 6a \rightarrow a = \frac{1}{6}$

$3 = 12a + 2b = 2 + 2b \rightarrow b = \frac{1}{2}$

$3 = 7a + 3b + c = \frac{7}{6} + \frac{3}{2} + c \rightarrow c = \frac{1}{3}$

$1 = a + b + c + d = \frac{1}{6} + \frac{1}{2} + \frac{1}{3} + d \rightarrow d = 0$

Ergebnis: $Tn = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n = \frac{1}{6}n(n^2 + 3n + 2) = \frac{1}{6}n(n + 1)(n + 2)$

Test: $T1 = \frac{1}{6}\cdot 1\cdot(1 + 1)(1 + 2) = 1$

$T2 = \frac{1}{6}\cdot 2\cdot(2 + 1)(2 + 2) = 4$

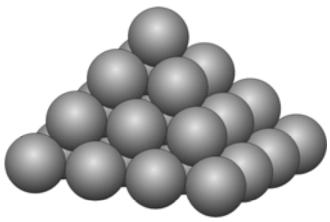
$T3 = \frac{1}{6}\cdot 3\cdot(3 + 1)(3 + 2) = 10$

$T4 = \frac{1}{6}\cdot 4\cdot(4 + 1)(4 + 2) = 20$

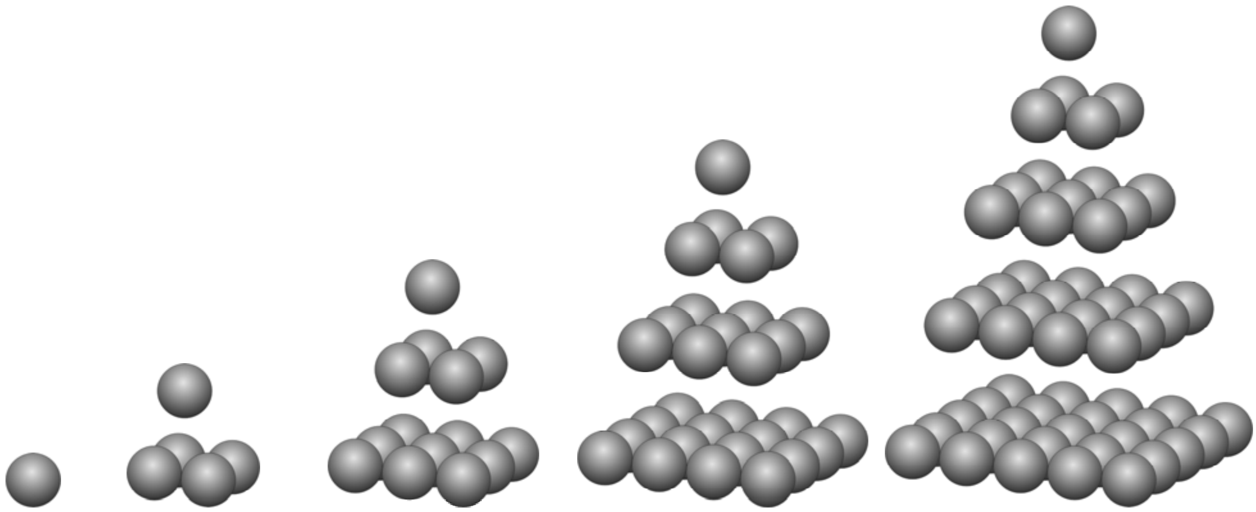
$T5 = \frac{1}{6}\cdot 5\cdot(5 + 1)(5 + 2) = 35$

$T6 = \frac{1}{6}\cdot 6\cdot(6 + 1)(6 + 2) = 56$

…



Pyramidenzahlen P1, P2, P3, … Pn



$P1 = 1$ $P2 = P1 + 2^2 = 5$ $P3 = P2 + 3^2 = 14$ $P4 = P3 + 4^2 = 30$ $P5 = P4 + 5^2 = 55$

Berechnung von Pn:

n =	1	2	3	4	5
Pn =	1	5	14	30	55
1. Differenz	4	9	16	25	36
2. Differenz	5	7	9	11	
3. Differenz	2	2	2	2	
4. Differenz	0	0	0		

Man erkennt:

$2 = 6a \rightarrow a = \frac{1}{3}$

$5 = 12a + 2b = 4 + 2b \rightarrow b = \frac{1}{2}$

$4 = 7a + 3b + c = \frac{7}{3} + \frac{3}{2} + c \rightarrow c = \frac{1}{6}$

$1 = a + b + c + d = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} + d \rightarrow d = 0$

Ergebnis: $Pn = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(2n^2 + 3n + 1) = \frac{1}{6}n(n + 1)(2n + 1)$